Int. J. Solids Structures, 1967, Vol. 3, pp. 799 to 808. Pergamon Press Ltd. Printed in Great Britain

TORSION OF ELASTIC CYLINDERS IN CONTACT

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Abstract—The torsion of two finite, coaxial, circular cylinders, of dissimilar materials, in contact, is considered. The surface of contact is assumed to consist of two parts: an exterior region of slip and an interior region of adhesion. Assuming that slip has progressed a sufficient amount, an analytical expression is given relating the angle of twist with the constant shear stress (assumed) in the slip region, necessary to eliminate a singularity in stress. The special case of an external crack, corresponding to zero shear stress in the slip region, is also discussed.

1. INTRODUCTION

THIS study considers the torsion of two finite coaxial circular cylinders, radius a, in contact. One end of the two cylinder system is held fixed while the other is given a rigid rotation about its axis through an angle, γ. Two situations are considered herein: the case where a sliding mechanism is allowed, e.g. as in the case of coulomb friction, and the case where the cylinders are bonded together on a coaxial region of radius $a_0 < a$.

A discussion of the analogous half-space problem by Mindlin [1] points out the need for a theory of slip when considering the torsional contact problem for spheres. The problem for the torsion of spheres incorporating this idea was subsequently solved by Lubkin [2] who used techniques appropriate to the half-space in potential theory. His method was to prescribe the following boundary conditions on the surface of the half-space:

$$\begin{aligned} v &= \alpha r & 0 \le r \le a_0 \\ \sigma_{z\theta}/\mu &= g(r) & a_0 \le r \le a \\ &= 0 & a \le r < \infty \end{aligned}$$

where a is the radius of contact, a_0 is the radius of the circle separating the slip region from the region of adhesion and $g(r)$ is a function describing the relationship between the normal force and the tangential shearing force. For Lubkin's problem, $g(r)$ was taken to be proportional to the normal stress as given by the Hertz theory. A later paper by Keer [3] considered the analogous problem of the torsion of a rigid die indenting an elastic layer.

The problem of two cylinders in contact reduces to the same considerations as in the half-space problem. If two cylinders bonded together by a circular region, radius $a_0 < a$, are then twisted, a stress singularity will occur for $r = a_0$. If a friction law is assumed for $r > a_0$, then the radius of contact, a_0, can be chosen so as to make the singularity vanish.

In the work to follow it is assumed that the law of friction produces a constant stress distribution, $\sigma_{z\theta}$, in the region of slip.

The next sections consider the formulation for the contact problem of two elastic, isotropic, and homogeneous cylinders, identical in geometry but of different materials. Boundary conditions analogous to the half-space problem are established and the solution to the problem will be seen to depend upon the solution to dual series equations that have been studied by Srivistav [4]. The solution of Srivistav is then applied to the particular problem at hand. An approximate result valid for $a_0/a < 0{\cdot}5$ is derived for the relation between angle of twist and the stresses imposed by the sliding friction necessary to eliminate a singularity in stress. Finally, the case of two cylinders bonded together by a circular region of radius $a_0 < a$ is discussed. As no friction will be assumed for this case, the "contact" problem is reduced to the problem of an external crack between two cylinders. The problem can be shown to be the same in this case, except for certain constants, as that treated by Sneddon *et al.* [5]. Griffith's theory of fracture [6, 7] is applied to this problem to determine the critical value of applied torque. The solutions to these problems will be sought by methods appropriate to the classical infinitesimal theory of elasticity.

2. DERIVATION OF EQUATIONS AND REDUCTION TO DUAL SERIES

The cylindrical coordinates r, θ and z are used, and the coordinate system is centered at the fixed end of the two cylinder system ($z = 0$, $0 \leq r \leq a$). The cylinder that is held fixed at one end occupies the region $0 \leq z \leq \delta$, $0 \leq r \leq a$, and is denoted by the superscript 1. The rotation, γ, is applied to the end ($z = 2\delta$, $0 \leq r \leq a$) of the second cylinder which occupies the region $\delta \leq z \leq 2\delta, 0 \leq r \leq a$, and is denoted by the superscript 2. Since only torsional stresses are set up in the cylinders when the rotation is applied, the only non-zero displacement is the circumferential displacement ω_θ. Further, the only non-zero components of the stress tensor are the shearing stresses

$$\sigma_{z\theta} = \mu\frac{\partial\omega_\theta}{\partial z}, \qquad \sigma_{r\theta} = \mu r\frac{\partial}{\partial r}\left(\frac{\omega_\theta}{r}\right) \tag{2.1}$$

where μ is the shear modulus. It is convenient to proceed with the solution in non-dimensional form by substituting

$$r = a\rho, \qquad z = ax, \qquad \omega_\theta = au_\theta, \qquad \sigma_{z\theta} = \mu S_{x\theta}, \qquad \sigma_{r\theta} = \mu S_{\rho\theta}.$$

In the absence of body forces, in the equilibrium state, u_θ satisfies the partial differential equation

$$\frac{\partial^2 u_\theta}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial u_\theta}{\partial\rho}-\frac{u_\theta}{\rho^2}+\frac{\partial^2 u_\theta}{\partial x^2} = 0. \tag{2.2}$$

The boundary conditions for the problem are:

$$\frac{\partial}{\partial\rho}\left(\frac{u_\theta^1}{\rho}\right) = 0, \qquad \rho = 1 \qquad \left(0 \leq x \leq \frac{\delta}{a}\right) \qquad \text{(curved surface of rod 1—stress free)} \tag{2.3}$$

$$\frac{\partial}{\partial\rho}\left(\frac{u_\theta^2}{\rho}\right) = 0, \qquad \rho = 1 \qquad \left(0 \leq x \leq \frac{\delta}{a}\right) \qquad \text{(curved surface of rod 2—stress free)} \tag{2.4}$$

$$u_\theta^1 = 0, \qquad x = 0 \qquad (0 \le \rho \le 1) \tag{2.5}$$

$$u_\theta^2 = \gamma\rho, \qquad x = 2\frac{\delta}{a} \qquad (0 \le \rho \le 1) \tag{2.6}$$

$$u_\theta^1 = u_\theta^2, \qquad x = \frac{\delta}{a} \qquad (0 \le \rho < c) \qquad \text{(continuity of } u_\theta) \tag{2.7}$$

$$\mu^1\frac{\partial u_\theta^1}{\partial x} = \mu^2\frac{\partial u_\theta^2}{\partial x}, \qquad x = \frac{\delta}{a} \qquad (0 < \rho \le 1) \qquad \text{(continuity of } \sigma_{z\theta}) \tag{2.8}$$

$$\mu^1\frac{\partial u_\theta^1}{\partial x} = \mu^2\frac{\partial u_\theta^2}{\partial x} = S, \qquad x = \frac{\delta}{a} \qquad (c < \rho \le 1) \tag{2.9}$$

where S is the constant stress distribution, $\sigma_{z\theta}$, in the region of slip, and $c = a_0/a$.

If the forms

$$u_\theta^1 = B_0\rho x + \sum_{n=1}^{\infty} B_n J_1(\lambda_n\rho)\sinh(\lambda_n x), \tag{2.10}$$

$$u_\theta^2 = \gamma\rho + A_0\rho\left(2\frac{\delta}{a} - x\right) + \sum_{n=1}^{\infty} A_n J_1(\lambda_n\rho)\sinh \lambda_n\left(2\frac{\delta}{a} - x\right) \tag{2.11}$$

are assumed, then the boundary conditions (2.3) and (2.4) are satisfied if the $\{\lambda_n\}$ are the positive zeros of the equation

$$\lambda J_1'(\lambda) - J_1(\lambda) = J_2(\lambda) = 0. \tag{2.12}$$

Further, the forms (2.10) and (2.11) satisfy boundary conditions (2.5) and (2.6), respectively.

Continuity of stress, boundary condition (2.8), requires that

$$A_0 = -\mu^1/\mu^2 B_0,$$

$$A_n = -\mu^1/\mu^2 B_n.$$

The displacement u_θ and stress $S_{x\theta}$ can now be written as

$$u_\theta^1 = B_0\rho x + \sum_{n=1}^{\infty} B_n J_1(\lambda_n\rho)\sinh(\lambda_n x), \tag{2.13}$$

$$u_\theta^2 = \gamma\rho - \mu^1/\mu^2 B_0\left(2\frac{\delta}{a} - x\right) - \mu^1/\mu^2 \sum_{n=1}^{\infty} B_n J_1(\lambda_n\rho)\sinh \lambda_n\left(2\frac{\delta}{a} - x\right), \tag{2.14}$$

$$S_{x\theta}^1 = B_0\rho + \sum_{n=1}^{\infty} \lambda_n B_n J_1(\lambda_n\rho)\cosh(\lambda_n x), \tag{2.15}$$

$$S_{x\theta}^2 = \mu^1/\mu^2 B_0\rho + \mu^1/\mu^2 \sum_{n=1}^{\infty} \lambda_n B_n J_1(\lambda_n\rho)\cosh \lambda_n\left(2\frac{\delta}{a} - x\right). \tag{2.16}$$

To satisfy boundary conditions (2.7) and (2.9), the following equations are obtained

$$(1 + \mu^1/\mu^2)B_0\rho\frac{\delta}{a} + (1 + \mu^1/\mu^2)\sum_{n=1}^{\infty} B_n J_1(\lambda_n\rho)\sinh\left(\lambda_n\frac{\delta}{a}\right) = \gamma\rho \qquad (0 \le \rho < c) \tag{2.17}$$

$$\mu_1 B_0\rho + \mu_1 \sum_{n=1}^{\infty} \lambda_n B_n J_1(\lambda_n\rho)\cosh\left(\lambda_n\frac{\delta}{a}\right) = S \qquad (c < \rho \le 1). \tag{2.18}$$

If the substitutions

$$\alpha = \delta/a, \qquad \beta = \gamma/(1+\mu^1/\mu^2), \qquad K = S/\mu^1, \qquad c_0 = B_0, \qquad c_n = \lambda_n B_n \cosh\left(\lambda_n \frac{\delta}{a}\right)$$

are made, and it is assumed that $\lambda_n(\delta/a)$ is sufficiently large* so that $\tanh[\lambda_n(\delta/a)] = 1$, equations (2.17) and (2.18) become

$$\alpha c_0 \rho + \sum_{n=1}^{\infty} \lambda_n^{-1} c_n J_1(\lambda_n \rho) = \beta\rho \qquad (0 \leq \rho < c) \tag{2.19}$$

$$c_0 \rho + \sum_{n=1}^{\infty} c_n J_1(\lambda_n \rho) = K \qquad (c < \rho \leq 1). \tag{2.20}$$

3. SOLUTION OF DUAL SERIES AND APPROXIMATION FOR LARGE SLIP AREA

The dual series equations (2.19) and (2.20) are special cases of the dual series studied by Srivistav [4]. Srivistav assumes that

$$c_0 \rho + \sum_{n=1}^{\infty} c_n J_1(\lambda_n \rho) = -\frac{\partial}{\partial \rho}\int_{\rho}^{c} \frac{g(t)\,\mathrm{d}t}{\sqrt{(t^2-\rho^2)}} \qquad (0 \leq \rho < c) \tag{3.1}$$

and then shows that the coefficients c_0 and c_n satisfy the relations

$$c_0 = 4\int_c^1 u^2 K\,\mathrm{d}u + 8\int_0^c u g(u)\,\mathrm{d}u, \tag{3.2}$$

$$c_n = \frac{2}{J_1^2(\lambda_n)}\left\{\int_c^1 u\beta u J_1(\lambda_n u) + \int_0^c \sin(\lambda_n u) g(u)\,\mathrm{d}u\right\}, \tag{3.3}$$

the function $g(t)$ satisfying the Fredholm integral equation

$$g(t) = \chi(t) + \int_0^c L(t,u) g(u)\,\mathrm{d}u, \tag{3.4}$$

where

$$\chi(t) = \frac{2}{\pi t}\int_0^t \frac{u^2 \beta u\,\mathrm{d}u}{\sqrt{(t^2-u^2)}} - \frac{16\alpha t}{\pi}\int_c^1 u^2 K\,\mathrm{d}u - \frac{2}{\pi}\int_c^1 uK\left\{2\sum_{n=1}^{\infty}\frac{J_1(\lambda_n u)\sin(\lambda_n t)}{\lambda_n J_1^2(\lambda_n)}\right\}\mathrm{d}u, \tag{3.5}$$

and

$$L(t,u) = \frac{16(1-2\alpha)}{\pi} tu + \frac{4}{\pi^2}\int_0^{\infty}\frac{K_2(y)}{I_2(y)}[8tuI_2(y) - \sinh(ty)\sinh(uy)]\,\mathrm{d}y. \tag{3.6}$$

* It is noted that $\lambda_n \geq \lambda_1 = 3{\cdot}8317$ and $\delta/a > 1{\cdot}4$ implies that $\tanh(\lambda_n \delta/a) = 1{\cdot}0000$.

Further, Srivistav shows that the infinite series in equation (3.5) can be written in the form

$$2\sum_{n=1}^{\infty}\frac{J_1(\lambda_n u)\sin(\lambda_n t)}{\lambda_n J_1^2(\lambda_n)} = \frac{2tH(u-t)}{u\sqrt{(u^2-t^2)}} - 4ut\sqrt{(1-t^2)} - \frac{2}{\pi}\int_0^{\infty}\frac{K_2(y)}{yI_2(y)}\sinh(ty)\{4uI_2(y)-yI_1(uy)\}\,\mathrm{d}y, \qquad (3.7)$$

where $H(\rho)$ is the Heaviside unit function.

If the expression for the infinite series given above is substituted into equation (3.5) and the integrations in each of the 3 terms in this equation are performed, Erdelyi [8], the free term, $\chi(t)$, becomes

$$\chi(t) = [4\beta/\pi - 16\alpha K(1-c^3)/3\pi]t - \frac{4K}{\pi}\left\{t\log\left(\frac{1+\sqrt{(1-t^2)}}{c+\sqrt{(c^2-t^2)}}\right) - \frac{2(1-c^3)}{3}t\sqrt{(1-t^2)} - \frac{1}{\pi}\int_0^{\infty}\frac{K_2(y)}{I_2(y)}\sinh(ty)F(c,y)\,\mathrm{d}y\right\}, \qquad (3.8)$$

where

$$F(c,y) = \left\{\frac{4(1-c^3)}{3}I_2(y) - \frac{\pi}{2}[L_1(y)I_0(y) - L_0(y)I_1(y) - cL_1(cy)I_0(cy) + cL_0(cy)I_1(cy)]\right\}. \qquad (3.9)$$

The functions L_0 and L_1 are modified Struve functions, McLachlan [9].

A numerical treatment, discussed in the sequel, indicated that for values of c less than 0·5, the infinite series portion of the kernel of the integral equation (3.4) can be neglected, in which case

$$g(t) = \chi(t) + \frac{16(1-2a)}{\pi}t\int_0^c ug(u)\,\mathrm{d}u, \qquad (3.10)$$

where $\chi(t)$ is given by equation (3.8). Hence

$$g(t) = \chi(t) + \frac{16(1-2\alpha)}{\pi}(AK + D\beta)t \qquad (3.11)$$

where

$$A = \frac{1}{K}\left[\int_0^c u\chi(u)\,\mathrm{d}u - \frac{4c^3\beta}{3\pi}\right] \Big/ \left[1 - \frac{16(1-2\alpha)c^3}{3\pi}\right], \qquad (3.12)$$

$$\int_0^c u\chi(u)\,\mathrm{d}u - \frac{4c^3\beta}{3\pi} = -\frac{4K}{\pi}\left\{\frac{4\alpha c^3(1-c^3)}{9} + \int_0^c u^2\log\left(\frac{1+\sqrt{(1-u^2)}}{c^2+\sqrt{(c^2-u^2)}}\right)\mathrm{d}u - \frac{(1-c^3)}{12}[\sin^{-1}c + c(1-c^2)^{\frac{1}{2}} - 2c(1-c^2)^{\frac{3}{2}}] - \frac{1}{\pi}\int_0^{\infty}\frac{K_2(y)}{I_2(y)}\left[\frac{cy\cosh(cy)-\sinh(cy)}{y^2}\right]F(c,y)\,\mathrm{d}y\right\}, \qquad (3.13)$$

and

$$D = \frac{4c^3}{3\pi} \bigg/ \left[1 - \frac{16(1-2\alpha)c^3}{3\pi}\right]. \tag{3.14}$$

The stress, $S_{x\theta}$, at the interface, given by equation (3.1), can be written in the alternate form

$$S_{x\theta}(\rho) = \frac{g(c)}{c\sqrt{(c^2-\rho^2)}}\rho - \rho\int_\rho^c \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{g(t)}{t}\right\}\frac{\mathrm{d}t}{\sqrt{(t^2-\rho^2)}} \qquad (0 \le \rho < c). \tag{3.15}$$

To investigate the stress at $\rho = c$, take $\rho = c-\varepsilon$ and change the variable of integration from t to $u = c-t$. It is easily seen that to the first order in ε

$$S_{x\theta}(c-\varepsilon) = \frac{g(c)}{\sqrt{(2c\varepsilon)}} + c\int_0^\varepsilon \frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{g(c-u)}{c-u}\right]\frac{\mathrm{d}u}{\sqrt{[2c(\varepsilon-u)]}}. \tag{3.16}$$

Now if $g(t)/t$ is differentiable in the neighborhood of $t = c$, this result can be written as

$$S_{x\theta}(c-\varepsilon) = \frac{g(c)}{\sqrt{(2c\varepsilon)}} + O(\sqrt{\varepsilon}).$$

Hence if $S_{x\theta}(c-\varepsilon)$ tends to a finite limit as $\varepsilon \to 0$, then

$$g(c) = 0. \tag{3.17}$$

For $c < 0{\cdot}5$, an approximate expression for $g(t)$ is given by equation (3.11). The criterion (3.17) is applied to this approximation to obtain the relation

$$\beta = K\left\{\frac{4(2\alpha-1)A - \bar{\chi}(c)}{[1-4(2\alpha-1)D]}\right\}, \tag{3.18}$$

where

$$\bar{\chi}(c) = \frac{\pi}{4cK}\left[\chi(c) - \frac{4c\beta}{\pi}\right], \tag{3.19}$$

and the terms within the braces are independent of K and β.

Equation (3.18) is a relationship involving the angle of twist, $\alpha = \delta/a$, and the constant shear stress in the slip region, necessary to eliminate a singularity in the shear stress, when $c < 0{\cdot}5$.

4. SPECIAL CASE OF AN EXTERNAL CRACK

In the case of an external crack, $K = 0$ and the free term given by equation (3.8) becomes

$$\chi(t) = \frac{4\beta}{\pi}t. \tag{4.1}$$

The integral equation (3.4) can now be written as

$$g(t) = \frac{4\beta}{\pi}t + \int_0^c L(t, u)g(u)\,\mathrm{d}u, \tag{4.2}$$

where the kernel, $L(t, u)$, is given by equation (3.6).

From equation (3.19) it immediately follows that

$$u_\theta' = \gamma\rho/(1+\mu^1/\mu^2) \qquad (x = \delta/a). \tag{4.3}$$

It is clear that the Reissner–Sagoci problem for a finite cylinder studied by Sneddon *et al.* [4] is recovered. If the two materials are unequal, quantities of physical interest, e.g. stress and displacement, will be functionally the same as in the Reissner–Sagoci problem. The resultant integral equation with the exception of certain modifications is the same as that given by Sneddon *et al.* and the solution is analogous.

The dependence on β and α of the governing integral equation (4.2) can be removed by a simple device. Define

$$G(t) = \frac{g(t)}{\dfrac{4\beta}{\pi}+\dfrac{16(1-2\alpha)}{\pi}\displaystyle\int_0^c ug(u)\,\mathrm{d}u}. \tag{4.4}$$

It is clear that $G(t)$ satisfies the integral equation

$$G(t) = t+\int_0^c K(t,u)G(u)\,\mathrm{d}u, \tag{4.5}$$

where

$$K(t,u) = \frac{4}{\pi^2}\int_0^\infty \frac{K_2(y)}{I_2(y)}[8tuI_2(y)-\sinh(ty)\sinh(uy)]\,\mathrm{d}y,$$

i.e. an equation independent of β and α.

A numerical solution of the function $G(t)$ was obtained by replacing the integral equation by a finite system of linear algebraic equations, following Kantorvich and Krylov [10]. The number of pivotal points ($G(t)$ determined at each pivotal point) on the interval $(0, c)$ was varied up to a maximum of 40 to investigate the smoothness and consistency of $G(t)$. Curves of the obtained values of $G(t)$ are given in Fig. 1 for $c = 0{\cdot}3, 0{\cdot}5, 0{\cdot}65, 0{\cdot}7, 0{\cdot}75, 0{\cdot}85$, and $0{\cdot}95$.

Multiplying both sides of equation (4.4) by t and integrating with respect to t from 0 to c, it is found that

$$g(t) = \frac{4\beta}{\pi}[1+\varphi/(1-\varphi)]G(t), \tag{4.6}$$

where

$$\varphi = \frac{16(1-2\alpha)}{\pi}\int_0^c uG(u)\,\mathrm{d}u.$$

Equation (4.6) is a convenient representation of the function $g(t)$ from which quantities of physical interest can be calculated.

For $c < 0{\cdot}5$, $G(t) \sim t$, and $g(t)$ is found to be

$$g(t) = \frac{4\beta}{\pi}\left[1+\frac{16(1-2\alpha)c^3/3\pi}{1-16(1-2\alpha)c^3/3\pi}\right]t. \tag{4.7}$$

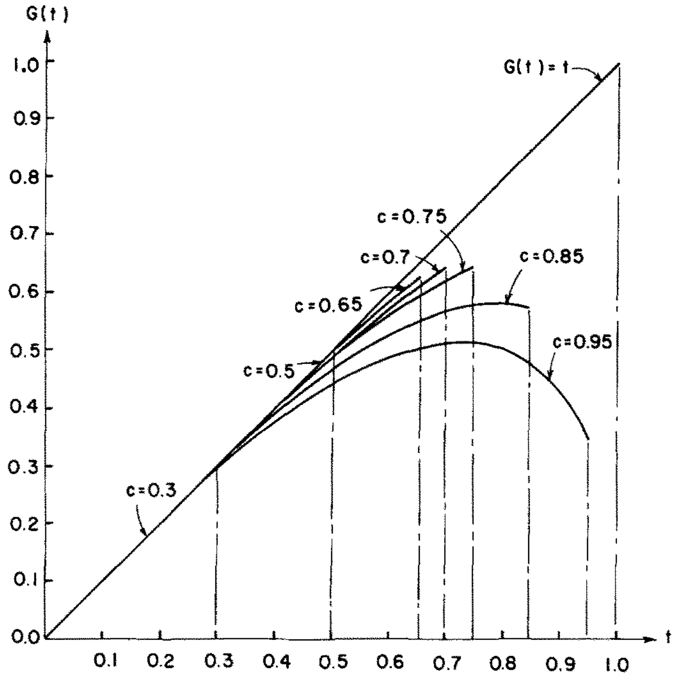

FIG. 1. $G(t)$ vs. t for $0 < c < 0{\cdot}95$.

The linearity of $g(t)$ for $c < 0{\cdot}5$ is a conclusion also reached by Sneddon *et al.* [5].

Using equation (3.13) for the shear stress at the interface of the two rods,

$$S_{x\theta}(\rho) = \frac{g(c)}{c}\frac{\rho}{\sqrt{(c^2-\rho^2)}}. \tag{4.8}$$

For points near c (i.e., $\rho \to c$), the stress $S_{x\theta}(c^-)$ is written as

$$S_{x\theta}(c^-) = \frac{N}{\sqrt{S}} \tag{4.9}$$

where $N = g(c)/\sqrt{(2c)}$ is called the stress intensity factor, Barenblatt [11], and S is the distance of a point in the interface from the contour $\rho = c$. A plot of N/β, for $\alpha = 10$, is given in Fig. 2. (For $c > 0{\cdot}5$, numerical integration was used to obtain values from equation (4.6).)

The torque M required to produce the twist, γ, is

$$M = 2\pi\mu^1 a^3\int_0^c \rho^2 S_{x\theta}\,\mathrm{d}\rho = \frac{4\pi c^3}{3}\left[\frac{g(c)}{c}\right],$$

or after some manipulation

$$M = \frac{\pi\mu^1 a^3\gamma}{2\alpha(1+\mu^1/\mu^2)}\left[1-\frac{3\pi-16c^3}{3\pi+16c^3(2\alpha-1)}\right], \qquad c < 0{\cdot}5. \tag{4.10}$$

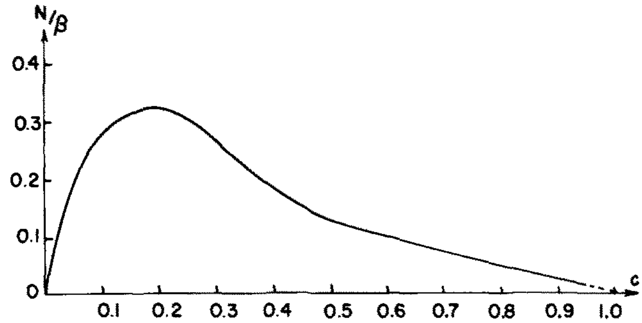

FIG. 2. Stress intensity factor N/β for $0 < c < 0{\cdot}95$.

The strain energy of the two cylinder system is

$$\tfrac{1}{2}M\gamma. \tag{4.11}$$

Expressing γ in terms of M from equation (4.10), the strain energy can be written in the form

$$\frac{(1+\mu^1/\mu^2)\alpha M^2}{\pi\mu^1 a^3}\left[1+\frac{3\pi-16c^3}{32\alpha c^3}\right]. \tag{4.12}$$

The increase in strain energy, due to the presence of the crack, is therefore

$$W = \frac{(1+\mu^1/\mu^2)M^2}{\pi\mu^1 a^3}\,\frac{3\pi-16c^3}{32c^3}. \tag{4.13}$$

It is noted that the strain energy, under the action of the applied torque M, is "increased" by the presence of the crack. This is in agreement with Spencer [12], where the change in strain energy of an elastic plate, in plane strain or plane stress, due to the presence of a crack, is considered.

The surface energy of the crack is

$$U = 2\pi(1-c^2)T_0, \tag{4.14}$$

where T_0 is the specific energy required to form the unit surface of the crack. The criterion for the crack to propagate is

$$\frac{\partial(U-W)}{\partial c} = 0. \tag{4.15}$$

Substituting equations (4.13) and (4.14) into (4.15), it is found that the critical value of applied torque is

$$M_{cr} = \frac{8}{3}\sqrt{\left[\frac{2\pi\mu^1 a^3 c^5 T_0}{(1+\mu^1/\mu^2)}\right]}, \qquad c < 0{\cdot}5. \tag{4.16}$$

Acknowledgements—In conclusion the authors wish to express their appreciation to the NASA Traineeship Program and to the U.S. Army Research Office-Durham for their financial support during the time in which this work was completed.

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(*Received* 2 *November* 1966)

Résumé—La torsion de deux cylindres circulaires co-axiaux finis, de matériaux dissemblables, en contact, est considérée. La surface de contact est supposée être constituée de deux parties: une région extérieure de glissement et une région intérieure d'adhésion. En supposant que le glissement a progressé d'une quantité suffisante, une expression analytique est donnée rapportant l'angle de torsion à l'effort de cisaillement (supposé) constant dans la région de glissement, nécessaire pour éliminer une particularité de tension. Le cas spécial d'une fissure externe, correspondant à un effort de cisaillement nul dans la région de glissement, est aussi discuté.

Zusammenfassung—Die Verdrehung zweier endlicher coaxialer kreisförmiger Zylinder aus ungleichen Materialien, im Kontakt wird untersucht. Die Kontaktflächen werden als zweiteilig angesehen, der äussere Gleitbereich und der innere Adhäsionsbereich. Vorausgesetzt, dass das Gleiten einen bestimmten Fortschritt gemacht hat, wird ein analytischer Ausdruck gegeben, der den Drehwinkel mit der angenommenen Spannung des Gleitbereiches gibt, der notwendig ist um Singularität der Spannung zu vermeiden. Der besondere Fall eines äusseren Risses, was einer Verschwindenden Scherspannung des Gleitbereiches entspricht, wird auch besprochen.

Абстракт—Исследуется кручение двух конечных, соосных, круглых цилиндров, изготовленых из разных материалов, находящихся в контакте. Подразумевается, что поверхность контакта состоит из двух частей: внешнего района скольжения и внутренного района сцепления. Предполагая, что скольжение имеет основное влияние, дается аналитическое выражение касающееся угла кручения с постоянным напряжением сдвига (предполагаемым) в районе скольжения, необходимым для устранения сингулярности в напряжении. Приводится также специальная задача внешней трещины, соответствующая нулевому напряжении сдвнга в районе скольжения.